A note on a piecewise-linear Duffing-type system

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Abstract

In [17] it was suggested that the number of limit cycles in a piecewise-linear system could be closely related to the number of zones, that is the number of parts of the phase plane where the system is linear. In this note we construct an example of a class of perturbed piecewise systems with n zones such that the first variation of the displacement function is identically zero. Then we conjecture that the system has no limit cycles using the second variation of the displacement function expressed for continuous functions. This system can be seen as a feedback system in control theory.

Keywords: Piecewise systems, limit cycles, Melnikov theory.

1 Introduction

In this work we consider a piecewise Duffing-type system. The classical Duffing system deals with continuous functions while the system addressed in this paper is a discontinuous one. This system can be seen as a feedback system in control theory. Such non-smooth systems appear naturally in many practical systems because many physical phenomena presents discontinuities. For example in control theory, systems controlled by switching belong to this class of non-smooth systems. It is known that switching occurs in control systems in industry such as multi-body systems, intelligent systems, robots [1]. Neuronal firing in biology [2] or impacts in mechanics are other systems which address the problem. The first works on this topic are [3] and [4]. Recently results can be found in [5], [6] and [7]. In some cases, piecewise-linear systems offer a good approximations for nonlinear complex systems offering a valuable tool for investigating nonlinear phenomena. As it was pointed out in [7], there is a *feeling* that piecewise-linear systems can present all the features met in nonlinear dynamics, such as homoclinic or heteroclinic orbits, limit cycles and attractors. In fact, Chua and collaborators discovered chaotic behavior in piecewise linear systems [9]. We will focus in the present work on the bifurcation phenomena related to the existence of invariant closed curves, such as limit cycles. The number and distribution (location) of limit cycles is one of the most important problems in qualitative theory of dynamical systems. There are numerous works on existence, number and distribution of limit cycles for continuous dynamical systems, for example [10, 11, 12, 13, 14, 15] but not many exist for non-smooth systems. Investigations of the dynamics of non-smooth systems with one, two or three lines of discontinuity are performed in [16], [5], [6]. Few papers deal with systems with a large number of discontinuities and not much is known about their dynamics. A recent work, which considers a piecewise-linear function for a

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Liénard system with n zones is [17]. In that work it is suggested that *the number of limit cycles is closely related to the number of zones.* In the present work, we construct a piecewise Duffing-type system with n zones and conjecture it has no limit cycles.

2   A piecewise-linear Duffing-type system

Assume H is a Hamilton function and consider the following perturbed differential system

$$\begin{aligned} \dot{x} &= \partial H/\partial y + \varepsilon f(x,y,\varepsilon) \\ \dot{y} &= -\partial H/\partial x + \varepsilon g(x,y,\varepsilon) \end{aligned} \tag{1}$$

where $f(x,y,\varepsilon)$ and $g(x,y,\varepsilon)$ are two enough smooth functions in x,y which depend analytically on a small parameter ε. For any h on a real interval (a,b), we suppose that the set $\{(x,y) \in \mathbb{R}; H(x,y) = h\}$ contains a closed curve C_h free of critical points (a circle for example) which depends continuously on h. Such a family of closed curves C_h corresponds to an annulus A of periodic solutions of the unperturbed Hamiltonian system $dH = 0$, that is, the system (1) for $\varepsilon = 0$.

Recall that, if we fix a transversal segment to the flow in (1) and parameterize it using the energy level h, then the function

$$d(h,\varepsilon) := P(h,\varepsilon) - h = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + ... + \varepsilon^k M_k(h) + O(\varepsilon^{k+1}), \ h \in (a,b), \tag{2}$$

where $P(h,\varepsilon)$ is the first return map (or Poincaré map), is the displacement function defined for small ε. The variations of the displacement function, $M_k(h)$, are also called the Melnikov functions. Computing explicitly the Melnikov functions is a challenging problem and as long as we know they are determined only in some particular cases. It is known that the number of zeros of the first non-disappearing $k-th$ order Melnikov function $M_k(h)$ provides the upper bound of the number of limit cycles of the perturbed system emerging from the periodic orbits of the unperturbed system [18]. More exactly, if the first not identically null Melnikov function is $M_k(h)$, then we have the following result:

THEOREM 2.1. *If $M_1(h) = ... = M_{k-1}(h) \equiv 0$, $M_k(h) \neq 0$ for some h and h_1 is a root of the Melnikov function $M_k(h)$ such that the $m-$derivative $M_k^{(m)}(h_1) \neq 0$, $m \geq 1$, then for $\varepsilon \neq 0$ sufficiently small, system (1) has one limit cycle of multiplicity m in an $O(\varepsilon)$ neighborhood of C_{h_1}. In case that $M_k(h) \neq 0$ for any h, then for $\varepsilon \neq 0$ sufficiently small, the system (1) has no limit cycles in an $O(\varepsilon)$ neighborhood of C_h.*

More details can be found in [21] and [17]. The first Melnikov function for the system (1), as it is reported in [22], is given by:

$$M_1(h) = \oint_{C(h)} g(x,y,0)dx - f(x,y,0)dy. \tag{3}$$

In [22] is presented a method for determining the second Melnikov function, following an algorithm described in [23], for a system of type (1) with the Hamiltonian expressed in the form $H(x,y) = \frac{1}{2}y^2 - U(x)$, where $U(x)$ is a polynomial of degree at least 2, in the case when the first Melnikov function $M_1(h) \equiv 0$. In this case, the second order Melnikov function is given by:

$$M_2(h) = \oint_{C_h} \frac{\partial g}{\partial \varepsilon}(x,y,0)dx - \frac{\partial f}{\partial \varepsilon}(x,y,0)dy, \tag{4}$$

provided that

$$\frac{\partial f}{\partial x}(x,y,0) + \frac{\partial g}{\partial y}(x,y,0) = 0.$$

For similar systems of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} + \varepsilon \begin{pmatrix} g_1(x,y,\varepsilon) \\ g_2(x,y,\varepsilon) \end{pmatrix}, \tag{5}$$

where $(x,y) \in \mathbb{R}^2$, $0 < \varepsilon \ll 1$ and $f = (f_1(x,y), f_2(x,y))$, $g = (g_1(x,y,\varepsilon), g_2(x,y,\varepsilon))$ are two sufficiently smooth functions, the first Melnikov function is treated in [21] (used in [17]) and is given by:

$$M_1(r) = \int_0^{T_r} e^{-\int_0^t div(f(\tau_r(s)))ds}(f_1(\tau_r(t))g_2(\tau_r(t),0) - g_1(\tau_r(t),0)f_2(\tau_r(t)))dt \tag{6}$$

where we assumed that the system (5) for $\varepsilon = 0$ possesses a family of periodic orbits $C_r : \tau_r(t), r > 0$ depending on a positive real parameter r and having the period T_r. This formula can be used even though f is differentiable but g is discontinuous in some isolated points, as remarked in [17]. It will be employed in the present work in such a case.

Consider now the planar discontinuous Duffing-type system given by:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ g(x,y,\varepsilon) \end{pmatrix}, \tag{7}$$

where g is a real function defined on the set $I \times \mathbb{R} \times (-\varepsilon_1, \varepsilon_1)$, with $I = (-\infty, a_1] \cup (a_1, a_2] \cup ... \cup (a_n, \infty)$, $a_i \in \mathbb{R}, i = 1, 2, ..., n$, $\varepsilon_1 > 0$, discontinuous in the points a_i and, for a fixed ε, differentiable with respect to (x,y) on each of the $(n+1)$ strips (a_i, a_{i+1}) where it is defined. If $g(x,y,\varepsilon) = x^3$ and $\varepsilon = \beta$ we meet the classical Duffing system [19] in a particular case. In [20] we studied a Duffing continuous system identifying the conditions to transition to chaos.

Our first result from this paper is stated in the following Proposition:

PROPOSITION 2.1. *Consider two increasing sequences of real numbers*
$a_0 = -\infty < 0 < a_1 < ... < a_n < a_{n+1} = +\infty,\ -\infty < \alpha_0 < \alpha_1 < ... < \alpha_n < +\infty,\ n \in \mathbb{N}, n > 1$ *and the non-smooth linear function with $n+1$ zones*

$$g(x,y,\varepsilon) = \alpha_i x + \varepsilon y \quad if \ x \in (a_i, a_{i+1}],\ i = 0, 1, ..., n \tag{8}$$

with the convention that the last interval is (a_n, a_{n+1}).

Then, for any $0 < \varepsilon \ll 1$ and any two sequences $(a_i)_{i=0,1,...,n+1}$ and $(\alpha_i)_{i=0,1,...,n}$ as above, the first Melnikov function of the system (7) is identically zero.

PROOF It is clear that the system (7) is of type (5) with $f_1 = y, f_2 = -x, g_1 = 0$ and $g_2(x,y,\varepsilon) = g(x,y,\varepsilon) = \alpha_i x + \varepsilon y$. With these functions, the unperturbed system (7) has a one-parameter family of periodic solutions which are circles of the form $C_r : x^2 + y^2 = r^2$. Choosing the parametrization $\tau_r(t) : x = r \sin t,\ y = r \cos t,\ t \in [0, 2\pi)$, and denoting $g(x) := g_2(x,y,0) = \alpha_i x$, then using (6) we get

$$M_1(r) = \int_0^{2\pi} r \cos t\ g(r \sin t)dt. \tag{9}$$

3

One can observe that the same result is recovered from the formula (3), at least formally since g is discontinuous, applied for systems of the form (1) with $H(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$, $f(x,y,\varepsilon) = 0$ and $g(x,y,\varepsilon) = \alpha_i x + \varepsilon y$, denoting the circle $C_r : x^2 + y^2 = r^2$ and $g(x) := g(x,y,0) = \alpha_i x$. Indeed, from (3) we have

$$M_1(r) = \oint_{C_r} g(x)dx, \tag{10}$$

and using the same parametrization $x = r\sin t$, $y = r\cos t$, $t \in [0, 2\pi)$, it leads to (9).

Find now $M_1(r)$. From (9) we have

$$M_1(r) = \int_0^{2\pi} r\cos t \; g(r\sin t)dt \tag{11}$$

$$= \int_0^{\pi/2} r\cos t \; g(r\sin t)dt + \int_{\pi/2}^{3\pi/2} r\cos t \; g(r\sin t)dt + \int_{3\pi/2}^{2\pi} r\cos t \; g(r\sin t)dt.$$

Compute in the following the three integrals. Let $m \in \{0, 1, ..., n\}$ such that $a_m < r < a_{m+1}$ and $t_0 = 0 < t_1 < t_2 < ... < t_m < t_{m+1} = \pi/2$ an increasing sequence of real numbers in $[0, \pi/2]$ given by $\sin t_i = \frac{a_i}{r}$, $i = 1, 2, ..., m$. It is obvious now that if $t \in (t_i, t_{i+1})$, then $r\sin t \in (a_i, a_{i+1})$, $i = 0, 1, 2, ..., m$. Consequently

$$M^1(r) := \int_0^{\pi/2} r\cos t \; g(r\sin t)dt = \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} r^2\alpha_i \cos t \sin t dt \tag{12}$$

$$= \frac{r^2}{2} \sum_{i=0}^{m} \alpha_i(\sin^2 t_{i+1} - \sin^2 t_i) \tag{13}$$

$$= \frac{r^2}{2}\alpha_m - \frac{1}{2} \sum_{i=1}^{m} (\alpha_i - \alpha_{i-1})a_i^2.$$

For the second integral, consider a decreasing sequence of real numbers $t_{m+1} = \pi/2 < t_m < t_{m-1} < ... < t_1 < t_0 = 3\pi/2$ given by $\sin t_i = \frac{a_i}{r}$, $i = 1, 2, ..., m$. Because the sin function is decreasing on the interval $[\pi/2, 3\pi/2]$, one gets that, if $t \in (t_{i+1}, t_i)$, then $r\sin t \in (a_i, a_{i+1}), i = 0, 1, 2, ..., m$. Therefore

$$M^2(r) := \int_{\pi/2}^{3\pi/2} r\cos t \; g(r\sin t)dt = \sum_{i=0}^{m} \int_{t_{i+1}}^{t_i} r^2\alpha_i \cos t \sin t dt \tag{14}$$

$$= \frac{r^2}{2} \sum_{i=0}^{m} \alpha_i(\sin^2 t_i - \sin^2 t_{i+1}) \tag{15}$$

$$= \frac{r^2}{2}(\alpha_0 - \alpha_m) + \frac{1}{2} \sum_{i=1}^{m} (\alpha_i - \alpha_{i-1})a_i^2.$$

In the last case, because $r\sin t < 0$ for $t \in [3\pi/2, 2\pi)$ we have that $r\sin t \in (a_0, a_1)$ so that $g(r\sin t) = \alpha_0 r\sin t$.
Therefore

4

$$M^3(r) := \int_{3\pi/2}^{2\pi} r\cos t \; g(r\sin t)dt = \int_{3\pi/2}^{2\pi} r^2\alpha_0 \cos t \sin t dt = -\frac{r^2}{2}\alpha_0. \qquad (16)$$

Finally,

$$M_1(r) = M^1(r) + M^2(r) + M^3(r) = 0. \qquad (17)$$

$\blacksquare$

In the following we consider a much more general class of discontinuous functions

$$g(x, y, \varepsilon) = \alpha_i h'(x) + \varepsilon y \;\; if \; x \in (a_i, a_{i+1}], \;\; i = 0, 1, ..., n \qquad (18)$$

with the same convention for the last interval, i.e. it is (a_n, a_{n+1}), where $h'(x)$ is the derivative of a differentiable function $h(x)$ and we will prove a similar result given by:

PROPOSITION 2.2. *If g is a non-smooth function given by (18), then, for any $0 < \varepsilon \ll 1$ and any two sequences $(a_i)_{i=0,1,...,n+1}$ and $(\alpha_i)_{i=0,1,...,n}$ as above, the first Melnikov function of the system (7) is identically zero.*

The proof is similar. Proceeding as above, we have that:

$$M^1(r) := \int_0^{\pi/2} r\cos t \; g(r\sin t)dt = \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} r\alpha_i \cos t \; h'(r\sin t)dt \qquad (19)$$

$$= h(r)\alpha_m - h(0)\alpha_0 - \sum_{i=1}^{m}(\alpha_i - \alpha_{i-1})h(a_i),$$

$$M^2(r) := \int_0^{\pi/2} r\cos t \; g(r\sin t)dt = \sum_{i=0}^{m} \int_{t_{i+1}}^{t_i} r\alpha_i \cos t \; h'(r\sin t)dt \qquad (20)$$

$$= -h(r)\alpha_m + h(-r)\alpha_0 + \sum_{i=1}^{m}(\alpha_i - \alpha_{i-1})h(a_i),$$

and

$$M^3(r) := \int_{3\pi/2}^{2\pi} r\cos t \; g(r\sin t)dt = \int_{3\pi/2}^{2\pi} r\alpha_0 \cos t h'(r\sin t)dt = h(0)\alpha_0 - h(-r)\alpha_0, \qquad (21)$$

so we arrive to the same result:

$$M_1(r) = M^1(r) + M^2(r) + M^3(r) = 0. \qquad (22)$$

$\blacksquare$

5

As the first Melnikov function is identically zero, we can say nothing about the number of limit cycles. However, we conjecture the following fact:

CONJECTURE: *The non-smooth system (7) with the function g given by (18) has no limit cycles.*

We base this conjecture of the following fact. Compute the second Melnikov function for the system (7) with g given by (18). One can observe that $\frac{\partial f}{\partial x}(x, y, 0) + \frac{\partial g}{\partial y}(x, y, 0) = 0$ on any strip where g is defined, so using (4) we get that:

$$M_2(r) = \oint_{C_r} \frac{\partial g}{\partial \varepsilon}(x, y, 0) dx - \frac{\partial f}{\partial \varepsilon}(x, y, 0) dy = \oint_{C_r} y dx = \int_0^{2\pi} r^2 \cos^2 t \, dt = r^2 \pi \neq 0. \qquad (23)$$

As $M_2(r) = 0, r > 0$ has no root the above Conjecture is justified by Theorem 2.1. However, we can not present this Conjecture as a result, because the formula of the second Melnikov function which we used from [22] was proved for continuous functions but we applied it for functions with a finite number of discontinuities.

3 Conclusions

In this work we started to explore an example of a class of non-smooth dynamical systems with the first Melnikov function identically zero. We applied the formula of the second Melnikov function reported in a work of Iliev [22] in order to investigate further the existence of limit cycles. This formula has been deduced for continuous functions but it is quite plausible it remains valid for piecewise functions.

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